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Fluid Spheres and R- and T-Regions in General Relativity

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Abstract

R- and T-regions of spacetime are first defined in a particular coordinate system and then with the aid of the Schwarzschild vacuum solution are shown to represent the outside and inside of a black hole respectively. A certain class of interior solutions, relating to a perfect fluid, are also considered and it is found that these R- and T-solutions have distinct physical properties. The R-solutions are static, spherically symmetric, permanent, and have a classical analogue, while the corresponding T-solutions, which are wholly time dependent, are cylindrical, temporary, and do not have a classical analogue. It is shown that these T-solutions cannot be generated from their R-region counterparts. Particular T-solutions are also presented in which the corresponding fluid occupies the whole of a T-region. The fluid would under certain circumstances be black body radiation while for other cases the internal pressure is always greater than the density.

1. *Introduction*

It is well known that after a distribution of matter and energy has collapsed into a black hole an outside observer can receive no further information about the distribution, apart from the fact that it still possesses an external gravitational field. Since human scientists are presumably observers outside a black hole, they cannot discover what kind of physics prevails within it. The usual assumption made is that the same laws of physics, in particular Einstein's general relativity, apply within the hole as well as outside it. This assumption will be made in the following investigation.

One way of dealing with black holes is to employ the concept of the R-regions and T-regions of space-time, an idea first introduced by Novikov (1961) and later discussed in more detail by Zeldovich and Novikov (1971). In both works, however, the discussion is confined to the physical differences between the vacuum R-solution and T-solution of Einstein's equations. It seems that only in the case of Ruban (1968; 1969) are examples of interior T-solutions considered in any detail.

In this paper R- and T-regions will first be defined and then the vacuum cases will be considered briefly. The main discussion, however, centers on certain differing properties of the interior R- and T-solutions for material defined as a perfect fluid. Certain particular T-solutions will also be presented.

The distribution of material to be considered is initially supposed to be spherically symmetric around the origin. This means that an observer describing the physics of the configuration may use a general metric of the form:

$$\left.\begin{aligned} d\sigma^2 &= e^{2\lambda}\, d\eta^2 - e^{2\mu}\, d\xi^2 - r^2\, d\Omega^2 \\ d\Omega^2 &= \sin^2\theta\, d\phi^2 + d\theta^2 \end{aligned}\right\} \tag{1.1}$$

where λ, μ and r are functions of both the radial coordinate ξ and time coordinate η. The variables ξ and η have been chosen such that the speed of light c and the Newtonian constant of gravitation are both unity. The functions λ, μ and r for a given problem, can be determined from Einstein's equations when the nature of the energy-momentum tensor is known. In order to introduce the concept of the R- and T-region consider an orthogonal transformation of coordinates of the type

$$\Sigma = \Sigma\,(\xi, \eta), \qquad \Pi = \Pi(\xi, \eta) \tag{1.2}$$

where Σ is a new space-like coordinate, Π is a time-like coordinate and θ, ϕ remain unaltered. With the identifications $\Sigma \equiv x^1$, $\Pi \equiv x^4$, $\xi \equiv \bar{x}^1$, $\eta \equiv \bar{x}^4$, $\theta \equiv x^2 = \bar{x}^2$, $\phi \equiv x^3 = \bar{x}^3$ the metric (1.1) is transformed into

$$d\sigma^2 = \frac{e^{4\lambda}\dot{\Sigma}^2\, d\Pi^2}{\Pi'^2(\Sigma'^2 e^{2\lambda} - \dot{\Sigma}^2 e^{2\mu})} - \frac{e^{2(\lambda+\mu)}\, d\Sigma^2}{(\Sigma'^2 e^{2\lambda} - \dot{\Sigma}^2 e^{2\mu})} - r^2\, d\Omega^2 \tag{1.3}$$

where a prime means $\partial/\partial\xi$ and a dot, $\partial/\partial\eta$, and the coefficients of $d\Pi^2$, $d\Sigma^2$ and $d\Omega^2$ are expressed in terms of Σ and Π. The condition expressing orthogonality is

$$e^{2\mu}\dot{\Sigma}\dot{\Pi} - e^{2\lambda}\Sigma'\Pi' = 0 \tag{1.4}$$

and moreover, it will be assumed in (1.3) that

$$\Sigma'^2 e^{2\lambda} - \dot{\Sigma}^2 e^{2\mu} > 0 \tag{1.5}$$

for all η and ξ, so as not to violate the condition that Σ is space-like and Π is time-like.

There are of course many ways of defining the functions Σ and Π. For example, it may be required that the coordinates in the metric (1.3) be isotropic. This condition, with equation (1.4), would give rise to two coupled differential equations for the transformation functions. In this paper two possible alternative frames of reference are examined and these lead to the concept of the R-region and the T-region. It is, of course, necessary in order to complete the foregoing theory that the components of the Einstein tensor associated with (1.1) should also be transformed. However, this is unnecessary for the present discussion.

Definition of an R-region and a T-region. An event $(\Sigma, \theta, \phi, \Pi)$, where Σ is space-like and Π is time-like, decribed by the metric (1.3) with (1.5), is said to occur in $\left\{\begin{matrix}\text{an R-region}\\ \text{a T-region}\end{matrix}\right\}$ of space-time if the function $r(\Sigma, \Pi)$ is such that $\left\{\begin{matrix} r = \Sigma \\ r = \Pi \end{matrix}\right\}$.

With the above definitions it seems that the notion of the R-and T-region only applies for particular coordinate systems, i.e. those for which $r = \Sigma$ and $r = \Pi$. However, in the following the definition will be generalized so as to include other frames of reference.

2. *R- and T-Regions for a Vacuum*

In order to show the distinction between these two regions, consider the situation in which a material sphere is completely submerged within the Schwarzschild horizon surface $\Sigma = 2\bar{m}$, so that a vacuum exists both inside and outside this surface. This is a black hole in which the material does not extend to the horizon. The gravitational field of the outer region would then be described by the Schwarzschild metric

$$d\sigma^2 = (1 - 2\bar{m}/\Sigma)\, d\Pi^2 - \frac{d\Sigma^2}{(1 - 2\bar{m}/\Sigma)} - \Sigma^2\, d\Omega^2 \tag{2.1}$$

where $\bar{m}$ is a positive constant and Σ is space-like. Therefore, (2.1) describes an R-region only for $\Sigma > 2\bar{m}$.

With regard to the inner vacuum field the argument of Novikov (1961) will be employed. He assumes that the R-solution (2.1) is also valid in the T-region $(\Sigma < 2\bar{m})$ provided that it is written in the form:

$$d\sigma^2 = -\left(\frac{2\bar{m}}{\Sigma} - 1\right) d\Pi^2 + \frac{d\Sigma^2}{\left(\frac{2\bar{m}}{\Sigma} - 1\right)} - \Sigma^2\, d\Omega^2 \tag{2.2}$$

and that it is assumed that Σ is now time-like and Π is space-like, The coordinates Σ, Π may be relabeled according to $\Sigma \equiv \tau$, $\Pi \equiv q$ so that (2.2) now reads:

$$d\sigma^2 = \frac{d\tau^2}{\left(\frac{2\bar{m}}{\tau} - 1\right)} - \left(\frac{2\bar{m}}{\tau} - 1\right) dq^2 - \tau^2\, d\Omega^2 \tag{2.3}$$

where τ is the time coordinate in the T-region and q is the space coordinate.

This procedure may be thought to beg the question of whether or not R-solutions can be converted into T-solutions simply by reinterpreting the meaning of the coordinates. However, a vacuum solution of Einstein's field equations can be found by assuming in (1.1) that $r = \eta$ and that λ, μ are functions of η

alone. With this metric, and a zero cosmical constant, Einstein's equations for a vacuum are:

$$0 = e^{-2\lambda}\left\{\frac{1}{\eta^2} - \frac{2\dot{\lambda}}{\eta}\right\} - \frac{1}{\eta^2},$$

$$0 = e^{-2\lambda}\left\{\frac{\dot{\mu}}{\eta} - \frac{\dot{\lambda}}{\eta} + \ddot{\mu} + \dot{\mu}^2 - \dot{\lambda}\dot{\mu}\right\},$$

$$0 = e^{-2\lambda}\left\{\frac{1}{\eta^2} + \frac{2\dot{\mu}}{\eta}\right\} + \frac{1}{\eta^2}$$

where $\dot{F}(\eta) = dF/d\eta$. It is easy to show that all three equations are satisfied by

$$e^{2\lambda} = e^{-2\mu} = 1/(A/\eta - 1)$$

where A is the only nontrivial constant of integration. Hence (2.3) follows if $\tau \equiv \eta$, $q \equiv \xi$ and if A is arbitrarily identified with $2\overline{m}$. This last identification is hardly satisfactory; it would be necessary to show that at the boundary $\Sigma = 2\overline{m}$ of an R-region it was possible to fit the vacuum T-solution to (2.1) and thus to determine A in terms of $2\overline{m}$. This problem has not been worked out here.

A T-region exhibits a number of strange properties as is also mentioned by Ruban (1968) and Korkina and Gladush (1972). For example, in (2.3), unlike (2.1), the time coordinate is bounded above by $\tau < 2\overline{m}$ which means that the T-region is only a temporary phenomenon. Furthermore, from the point of view of an R-observer the horizon occurs for a specific value of the radial coordinate (namely $\Sigma = 2\overline{m}$) while for a T-observer it occurs when $\tau = 2\overline{m}$, that is for a particular finite value of the time. In other words the two observers interpret the horizon in completely different ways. Moreover, the surfaces τ = constant do not exhibit spherical symmetry; these surfaces are better described as hypercylinders. This is in contrast to the surfaces Π = constant in an R-region which are hyperspheres. Yet a further feature of the vacuum T-solution can be obtained when the motion of photons is considered. From (2.3) the outward radial coordinate speed of a photon is given by

$$\frac{dq}{d\tau} = \frac{1}{(2\overline{m}/\tau - 1)} \tag{2.4}$$

and so upon integration

$$ae^q = \frac{e^{-\tau}}{(2\overline{m} - \tau)^{2\overline{m}}} \tag{2.5}$$

where a is an arbitrary positive constant of integration. Therefore $q \to \infty$ as $\tau \to 2\overline{m}$, which presumably means that after a finite time $2\overline{m}$ the photon will have traveled an infinite value of the q coordinate.

In conclusion therefore it seems to be the case that the horizon occurring in (2.1) and (2.3) is the dividing surface between two substantially different regions.

3. *Interior R-Solutions*

In this Section R-regions are considered with particular reference to interior solutions of Einstein's equations with zero cosmical constant, Λ.

A useful device in the following is the mass function (Cahill and McVittie, 1970). For the metric (1.1) this is given by

$$m(\xi, \eta) = \tfrac{1}{2} r\{1 + e^{-2\lambda}\dot{r}^2 - e^{-2\mu} r'^2\} \tag{3.1}$$

where $\dot{r} = \partial r/\partial \eta$ and $r' = \partial r/\partial \xi$. This function remains invariant under transformations of the type (1.2) and so for the metric (1.3) in terms of Σ and Π

$$\frac{2m(\Sigma, \Pi)}{r} - 1 = e^{-2(\lambda+\mu)}\{e^{+2\mu}\dot{\Sigma}^2 - e^{2\lambda}\Sigma'^2\}\left\{r_\Sigma{}^2 - \frac{r_\Pi{}^2 \Pi'^2}{\dot{\Sigma}^2 e^{2(\mu-\lambda)}}\right\} \tag{3.2}$$

where $r_\Sigma = \partial r/\partial \Sigma$, $r_\Pi = \partial r/\partial \Pi$.

In an R-region for which $r = \Sigma$, equation (3.2) reduces to

$$1 - \frac{2m(\Sigma, \Pi)}{\Sigma} = \frac{e^{2\lambda}\Sigma'^2 - e^{2\mu}\dot{\Sigma}^2}{e^{2(\lambda+\mu)}} \tag{3.3}$$

and so (1.3) can be written as

$$\left.\begin{aligned} d\sigma^2 &= e^{2X}\, d\Pi^2 - e^{2Y}\, d\Sigma^2 - \Sigma^2\, d\Omega^2 \\ e^{2X} &= \frac{e^{2(\mu-\lambda)}\dot{\Sigma}^2}{\Pi'^2\left\{1 - \dfrac{2m(\Sigma, \Pi)}{\Sigma}\right\}}, \qquad e^{-2Y} = 1 - \frac{2m(\Sigma, \Pi)}{\Sigma} \end{aligned}\right\} \tag{3.4}$$

where X, Y are functions of Σ, Π in general. Therefore, the metric (3.4) will apply in an R-region provided that

$$1 - \frac{2m(\Sigma, \Pi)}{\Sigma} > 0 \tag{3.5}$$

for all Σ, Π.

It will be assumed that the interior R-solution can be fitted to an exterior vacuum solution whose metric is of the form (2.1) and will be written

$$d\sigma^2 = (1 - 2\overline{m}/\Sigma_e)\, d\Pi_e{}^2 - (1 - 2\overline{m}/\Sigma_e)^{-1}\, d\Sigma_e{}^2 - \Sigma_e{}^2\, d\Omega^2 \tag{3.6}$$

It will be sufficient to consider the continuity at the boundary of the coefficients of $d\Omega^2$ and of $d\Sigma_e{}^2$, $d\Sigma^2$ in (3.6) and (3.4). These give, respectively,

$$(\Sigma_e)_b = \Sigma_b, \qquad \frac{2m(\Sigma_b, \Pi)}{\Sigma_b} = \frac{2\overline{m}}{(\Sigma_e)_b}$$

where

$$2m(\Sigma_b, \Pi) = 2\bar{m} \tag{3.7}$$

This equation gives the boundary value of Σ, or of Σ_e, at each instant of Π-time. However, there is a limit to the validity of (3.7) because the minimum value of $(\Sigma_e)_b$ is $2\bar{m}$, if (3.6) is to be an R-solution. Hence (3.4) is an R-solution up to (or from) an instant Π_L given by the solution of the algebraic equation

$$2m(2\bar{m}, \Pi_L) = 2\bar{m} \tag{3.8}$$

Physically, of course, the limitation occurs only when Π_L in (3.8) is real; clearly there may also be more than one real value of Π_L, according to the nature of the function m.

For the subsequent comparison of R- and T-solutions, it is useful to consider the R-solutions for a static perfect fluid. This means that in (3.4)

$$X = X(\Sigma), \qquad Y = Y(\Sigma) \tag{3.9}$$

Components of the Einstein tensor for the metric (3.4) will be denoted by $G_b{}^a$ $(a, b = 1, 2, 3, 4)$ with the coordinate identifications $x^1 = \Sigma, x^2 = \theta$, $x^3 = \phi, x^4 = \Pi$. The Einstein equations for the statical fluid case then reduce to

$$8\pi\rho = -G_4{}^4 = \frac{1}{\Sigma^2} - e^{-2Y}\left\{\frac{1}{\Sigma^2} - \frac{2Y_\Sigma}{\Sigma}\right\}, \tag{3.10}$$

$$8\pi p = G_1{}^1 = e^{-2Y}\left\{\frac{1}{\Sigma^2} + \frac{2X_\Sigma}{\Sigma}\right\} - \frac{1}{\Sigma^2}, \tag{3.11}$$

$$8\pi p = G_2{}^2 = e^{-2Y}\left\{\frac{X_\Sigma - Y_\Sigma}{\Sigma} + X_{\Sigma\Sigma} + X_\Sigma{}^2 - X_\Sigma Y_\Sigma\right\} \tag{3.12}$$

where ρ, p are the density and pressure of the fluid, respectively, and a suffix Σ denotes $d/d\Sigma$. The last two equations give the condition for the isotropy of the pressure, namely,

$$X_{\Sigma\Sigma} + X_\Sigma{}^2 - X_\Sigma Y_\Sigma - \frac{X_\Sigma + Y_\Sigma}{\Sigma} + \frac{1}{\Sigma^2}(e^{2Y} - 1) = 0 \tag{3.13}$$

The equation (3.10) may be integrated to give

$$e^{-2Y} = 1 - \frac{8\pi\int\rho(\Sigma)\Sigma^2\, d\Sigma + C}{\Sigma} \tag{3.14}$$

where C is the constant of integration. Comparison with (3.4) then shows that

$$2m(\Sigma) = 8\pi\int\rho(\Sigma)\Sigma^2\, d\Sigma + C \tag{3.15}$$

The presence of a nonzero value of C permits the possibility of having $m(0) = 0$ even when $\rho(0)$ is unbounded, for example, when $\rho(\Sigma) = \Sigma^{-2}\exp\{-\Sigma/(8\pi C)\}$.

The vanishing of the vectorial divergence of the energy tensor yields the well known result

$$p_\Sigma = -\frac{\rho m(\Sigma)}{\Sigma^2}\left(1+\frac{p}{\rho}\right)\left\{1+\frac{4\pi p\Sigma^3}{m(\Sigma)}\right\}\left(1-\frac{2m(\Sigma)}{\Sigma}\right)^{-1} \tag{3.16}$$

This is the relativistic analogue of the Newtonian equation of hydrostatic support, since in the classical limit, it reduces to

$$p_\Sigma = -\frac{\rho m(\Sigma)}{\Sigma^2} \tag{3.17}$$

in which $m(\Sigma)$ is defined by (3.15) with $C = 0$ if, as is usually the case, $\rho(0)$ is a finite constant.

4. *Interior T-Solutions and their Relation to R-Solutions*

For a T-region $r = \Pi$ and so the mass function (3.2) reduces to:

$$\frac{2m^*(\Sigma, \Pi)}{\Pi} - 1 = \frac{\Pi'^2}{e^{4\mu}\dot{\Sigma}^2}(e^{2\lambda}\Sigma'^2 - e^{2\mu}\dot{\Sigma}^2) \tag{4.1}$$

where the asterisk is used in $m^*(\Sigma, \Pi)$, and on other functions to distinguish them from their R-region counterparts. In this section the coordinates Π, Σ will be relabeled according to $\Pi \equiv \tau$, $\Sigma \equiv q$ to avoid confusion between R- and T-solutions. Therefore, with this relabeling and (4.1), the metric (1.3) may be written in the following form:

$$\left.\begin{aligned} &d\sigma^2 = e^{2V}\,d\tau^2 - e^{2W}\,dq^2 - \tau^2\,d\Omega^2, \\ &e^{-2V} = \frac{2m^*(q,\tau)}{\tau} - 1, \qquad e^{2W} = \frac{e^{2(\lambda-\mu)}\tau'^2}{\dot{q}^2\left\{\dfrac{2m^*(q,\tau)}{\tau} - 1\right\}} \end{aligned}\right\} \tag{4.2}$$

where V, W are functions of τ and q alone. Hence (4.2) will apply in a T-region provided that

$$\frac{2m^*(q,\tau)}{\tau} - 1 > 0 \tag{4.3}$$

It will be assumed that the interior T-solution (4.2) is fitted to an exterior vacuum T-solution with a metric of form (2.3). By an argument analogous to that which established (3.7) and (3.8) it follows that the junction of the two metrics occurs at the instant τ_b of the internal time τ, and that (3.7), (3.8) are replaced, respectively, by

$$2m^*(\tau_b, q) = 2\bar{m} \tag{4.4}$$

$$2m^*(2\bar{m}, q_L) = 2\bar{m} \tag{4.5}$$

The first equation is interpretable as giving the value of q corresponding to the instant τ_b. The second (algebraic) equation gives the value of q_L corresponding to the choice of the maximum possible value of the τ-time employed in (2.3) as the moment at which the junction is made.

The components of the Einstein tensor for the metric (4.2) are denoted by G_b^{*a} $(a, b = 1, 2, 3, 4)$ with the coordinate identifications $x^1 = q, x^2 = \theta$, $x^3 = \phi, x^4 = \tau$. Consider the case of a perfect fluid of density ρ^* pressure p^* when

$$V = V(\tau), \qquad W = W(\tau) \tag{4.6}$$

The Einstein equations reduce to the three equations

$$8\pi\rho^* = -G_4^{*4} = e^{-2V}\left\{\frac{1}{\tau^2} + \frac{2W_\tau}{\tau}\right\} + \frac{1}{\tau^2} \tag{4.7}$$

$$8\pi p^* = G_1^{*1} = -e^{-2V}\left\{\frac{1}{\tau^2} - \frac{2V_\tau}{\tau}\right\} - \frac{1}{\tau^2} \tag{4.8}$$

$$8\pi p^* = G_2^{*2} = -e^{-2V}\left\{\frac{W_\tau - V_\tau}{\tau} + W_{\tau\tau} + W_\tau^2 - V_\tau W_\tau\right\} \tag{4.9}$$

where suffix τ denotes $d/d\tau$. The condition for the isotropy of pressure is therefore

$$W_{\tau\tau} + W_\tau^2 - V_\tau W_\tau + \frac{V_\tau + W_\tau}{\tau} - (1 + e^{2V})\frac{1}{\tau^2} = 0 \tag{4.10}$$

The integral of (4.8) is

$$e^{-2V} = \frac{-8\pi\int p^*(\tau)\tau^2\, d\tau + C^*}{\tau} - 1 \tag{4.11}$$

where C^* is the constant of integration. Comparison with (4.2) shows that

$$2m^*(\tau) = C^* - 8\pi\int p^*(\tau)\tau^2\, d\tau \tag{4.12}$$

The vanishing of the vectorial divergence of G_b^{*a} leads to

$$\rho_\tau^* = -\frac{m^* p^*}{\tau^2}\left(1 + \frac{\rho^*}{p^*}\right)\left(\frac{4\pi\rho^*\tau^3}{m^*} - \frac{2\tau}{m^*} + 3\right)\left(\frac{2m^*}{\tau} - 1\right)^{-1} \tag{4.13}$$

an equation which does not appear to have a classical analogue.

The equations (4.4), (4.5) become, for internal T-solutions defined by (4.6),

$$2m^*(\tau_b) = 2\bar{m} \tag{4.14}$$

$$2m^*(2\bar{m}) = 2\bar{m} \tag{4.15}$$

The second of these equations provides a restriction on the constant C^* and on any other constants involved in $p^*(\tau)$. The first equation shows that the internal T-solution, like the external vacuum T-solution (2.3), is temporary. For if τ_e denotes the time in the external solution, then $(\tau_e)_b = \tau_b$ and it is known that (2.3) is valid only so long as $2\overline{m} \geqslant (\tau_e)_b$. Thus τ_b is limited in the same way.

The question now arises whether a T-solution can be obtained from an R-solution by the Novikov process which converted (2.1) into (2.3). In one sense this process may be regarded as the equivalent of moving all the events outside the horizon surface to the region within it. But in another sense the process can be looked upon as the assertion that the R-solution continues to be applicable in the region in which $\Sigma < 2m(\Sigma, \Pi)$ provided that the following coordinate transformation is made: Let the coordinates of (3.4) be denoted by

$$x^4 = \Pi, \qquad x^1 = \Sigma, \qquad x_2 = \theta, \qquad \phi = x^3 \tag{4.16}$$

and let the coordinate transformation be

$$x^{*1} = x^4, \qquad x^{*4} = x^1, \qquad x^{*2} = x^2, \qquad x^{*3} = x^3 \tag{4.17}$$

A covariant tensor $K_{\mu\nu}$ in the (x) system of (4.16) is converted to $K^*_{\mu\nu}$ in the (x^*) system by the usual formula

$$K^*_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^{*\mu}} \frac{\partial x^\beta}{\partial x^{*\nu}} K_{\alpha\beta}$$

in which the only nonzero partial derivatives are

$$\frac{\partial x^1}{\partial x^{*4}} = 1, \qquad \frac{\partial x^4}{\partial x^{*1}} = 1, \qquad \frac{\partial x^2}{\partial x^{*2}} = 1, \qquad \frac{\partial x^3}{\partial x^{*3}} = 1$$

The components therefore transform according to the scheme

$$\left.\begin{array}{llll} K^*_{44} = K_{11}, & K^*_{14} = K_{41}, & K^*_{41} = K_{14}, & K^*_{11} = K_{44}, \\ K^*_{4\nu} = K_{1\nu}, & K^*_{\nu 4} = K_{\nu 1}, & K^*_{1\nu} = K_{4\nu}, & K^*_{\nu 1} = K_{\nu 4}, (\nu = 2, 3) \\ & K^*_{\mu\nu} = K_{\mu\nu} \ (\mu, \nu = 2, 3) & & \end{array}\right\} \tag{4.18}$$

Thus, the coordinate transformation (4.17) has the effect of interchanging the indices 1 and 4 in the components $K_{\mu\nu}$, in which, of course, the (x) must also be replaced by the appropriate (x^*). The same rules apply to the transformation of a mixed tensor, $P_\nu{}^\mu$, as may easily be proved.

The foregoing process will now be applied to the metrical tensor $g_{\mu\nu}$ of (3.4) with, for brevity in notation, x^4, x^1 written as Π, Σ, respectively, while x^{*4}, x^{*1} are similarly written as τ, q. Since (1.4) is the statement that $g_{14} = g_{41} = 0$, it follows from (4.18) that $g^*_{14} = g^*_{41} = 0$ or that

$$e^{2\mu}\dot{\tau}\dot{q} - e^{2\lambda}\tau' q' = 0 \tag{4.19}$$

The component $g_{44} = e^{2X(\Pi,\Sigma)}$ and it transforms to

$$g^*_{11} = [e^{2X(\Pi,\Sigma)}]_{\Pi=q,\Sigma=\tau} = -\frac{e^{2(\lambda-\mu)}\tau'^2}{\dot{q}^2\{2m(q,\tau)/\tau - 1\}}$$

where (4.19) has also been used. Hence comparison with (4.2) shows that

$$g^*_{11} = -e^{2W(\tau,q)}$$

Again in (3.4)

$$g_{11} = -e^{2Y} = -\left\{1 - \frac{2m(\Sigma,\Pi)}{\Sigma}\right\}^{-1}$$

and this becomes

$$g^*_{44} = -\left\{1 - \frac{2m(\tau, q)}{\tau}\right\}^{-1}$$

and therefore comparison with (4.2) shows that

$$g^*_{44} \equiv e^{2V} = \left\{\frac{2m^*(\tau, q)}{\tau} - 1\right\}^{-1}$$

provided that m^* is the same function of (τ, q) as is m. Moreover, there are no concealed minus signs in g^*_{44}, g^*_{11} so long as (4.3) is satisfied.

The components g_{22}, g_{33} are converted by the identity transformation to g^*_{22}, g^*_{33} and so (3.4) has been turned into (4.2), in other words, the metric of an R-solution has been converted to that of a T-solution by the coordinate transformation (4.17).

But the conversion of the metrical tensor is not the end of the matter. The Einstein tensor becomes, under the transformation (4.17)

$$G_4^{*4} = G_1{}^1, \qquad G_1^{*1} = G_4{}^4, \qquad G_2^{*2} = G_2{}^2, \qquad G_3^{*3} = G_3{}^3$$

Thus in terms of ρ^*, p^* and ρ, p these equations give, respectively

$$\rho^* = -p(\tau, q), \qquad p^* = -\rho(\tau, q), \qquad p^* = +p(\tau, q) \tag{4.20}$$

and so the T-solution obtained in this way would have

$$\rho^* + p^* = 0 \tag{4.21}$$

and the R-solution from which it was derived would also have to have

$$\rho + p = 0 \tag{4.22}$$

Such solutions of Einstein's equations are usually rejected on physical grounds since both the density and the pressure are reckoned to be nonnegative. Another difficulty is that the isotropy of pressure in the R-solution is expressed by $G_1{}^1 = G_2{}^2$ which becomes $G_4^{*4} = G_2^{*2}$ under the transformation. This does not express the isotropy of pressure in the T-solution which Einstein's equations show to be $G_1^{*1} = G_2^{*2}$. Furthermore, if the R-solution is statical, so that its mass function is (3.15), then this mass-function can be transformed to (4.12) only if (4.21) and (4.22) are valid, and C is identified with C^*.

The conclusion is that the Novikov process does not transform an R- into a T-solution in a satisfactory manner. The two kinds of solution are best regarded as independent of one another. However, there is one exception, which occurs when $G_\nu{}^\mu$ and $G_\nu^{*\mu}$ are both null tensors. It is for this reason that the vacuum R-solution (2.2) could be transformed into the vacuum T-solution (2.3).

Consider now how the definition of an R-region and a T-region may be generalized so as to apply directly to the metric written in the form (1.1). In an R-region $r = \Sigma$ in (1.1) and

$$1 - \frac{2m(\Sigma, \Pi)}{\Sigma} > 0 \tag{4.23}$$

In a T-region the definition is $r = \tau$ and

$$\frac{2m^*(q, \tau)}{\tau} - 1 > 0 \tag{4.24}$$

It is known that under transformations of coordinates for which the spatial variables θ, ϕ remain unchanged, the mass function is invariant. Moreover, both Σ and τ are invariant under such transformations, since $-g_{22} = \Sigma^2, -g^*_{22} = \tau^2$ and these components of the metrical tensor are unaltered by the transformation.

Thus, by means of these invariant properties, R- and T-regions can be redefined so as to apply to the general metric (1.1). An event $(\xi, \theta, \phi, \eta)$ occurs in an R-region provided that

$$1 - \frac{2m(\xi, \eta)}{r(\xi, \eta)} > 0 \tag{4.25}$$

and in a T-region if

$$1 - \frac{2m(\xi, \eta)}{r(\xi, \eta)} < 0 \tag{4.26}$$

In both cases $m(\xi, \eta)$ is defined by (3.1) and $r(\xi, \eta) \geqslant 0$ for all ξ, η. Therefore, the conditions (4.25), (4.26) are convertible into those given by Zeldovich and Novikov (1971, equation (3.1.13)). The regions are separated by the surface

$$\frac{2m(\xi, \eta)}{r(\xi, \eta)} = 1$$

which is called the apparent horizon by Carr and Hawking (1974).

5. *An Interior Solution Occupying the Whole of a T-region*

Unlike the corresponding equation for the interior R-solution, solutions of the equation expressing isotropy of pressure (4.10) for a T-region are virtually unknown. Indeed the only interior T-model presented so far would seem to be that of a distribution of dust due to Ruban (1969). Another class of solutions is therefore presented here.

The solutions, which depend only on the variable τ, are obtained from (4.10) by arbitrarily imposing the definition:

$$e^W = \tau^A \tag{5.1}$$

where A is a constant. With (5.1) the condition expressing isotropy of pressure (4.10) reduces to

$$\frac{(A^2-1)}{\tau^2} + \frac{(1-A)V_\tau}{\tau} - \frac{e^{2V}}{\tau^2} = 0 \tag{5.2}$$

and thus there is no solution when $A = 1$. The two cases to be considered are:

(a) $1 + 2A \neq 0, \quad 1 + A \neq 0$; (b) $1 + 2A = 0$

The case $1 + A = 0$ is omitted because it was found to produce negative densities and pressures.

Case (a): $1 + 2A \neq 0, 1 + A \neq 0$. If equation (5.2) is solved it is found that

$$e^{-2V} = \frac{B}{\tau^{2(A+1)}} - \frac{1}{1-A^2} \tag{5.3}$$

where B is a constant. Hence the metric, density and pressure are, respectively,

$$d\sigma^2 = \frac{d\tau^2}{B\tau^{-2(A+1)} - (1-A^2)^{-1}} - \tau^{2A}\, dq^2 - \tau^2\, d\Omega^2 \tag{5.4}$$

$$\left.\begin{aligned} 8\pi\rho^* &= \frac{B(1+2A)}{\tau^{2(A+2)}} - \frac{A(A+2)}{(1-A^2)\tau^2} \\ 8\pi p^* &= \frac{B(1+2A)}{\tau^{2(A+2)}} + \frac{A^2}{(1-A^2)\tau^2} \end{aligned}\right\} \tag{5.5}$$

The boundary condition adopted is that the fluid occupies the whole of the T-region. This means that the interior T-solution is valid for all values of τ in the interval $0 \leqslant \tau \leqslant 2\bar{m}$, where $2\bar{m}$ is the constant in (2.1) or (2.3). A consequence of this condition is that the metric (5.4) will be singular when $\tau = 2\bar{m}$. In other words

$$\frac{B}{(2\bar{m})^{2(A+1)}} - \frac{1}{(1-A^2)} = 0 \tag{5.6}$$

Hence

$$B = \frac{(2\bar{m})^{2(A+1)}}{1-A^2} \tag{5.7}$$

and equations (5.4) and (5.5) now read:

$$d\sigma^2 = \frac{(1 - A^2)\, d\tau^2}{(2\bar{m}/\tau)^{2(A+1)} - 1} - \tau^{2A}\, dq^2 - \tau^2\, d\Omega^2 \tag{5.8}$$

$$\left.\begin{aligned} 8\pi\rho^* &= \frac{1}{\tau^2(1 - A^2)} \left\{(1 + 2A)\left(\frac{2\bar{m}}{\tau}\right)^{2(A+1)} - A(A + 2)\right\} \\ 8\pi p^* &= \frac{1}{\tau^2(1 - A^2)} \left\{(1 + 2A)\left(\frac{2\bar{m}}{\tau}\right)^{2(A+1)} + A^2\right\} \end{aligned}\right\} \tag{5.9}$$

Clearly, from the metric (5.8), $1 - A^2 > 0$ and so

$$\underset{\tau \to 0}{LT}\left(\frac{2\bar{m}}{\tau}\right)^{2(A+1)} = \infty$$

Therefore, the dominant term in the density and pressure as $\tau \to 0$ is

$$\frac{1 + 2A}{\tau^2(1 - A^2)}\left(\frac{2\bar{m}}{\tau}\right)^{2(A+1)}$$

and it must be positive. Thus again, $1 - A^2 > 0$, and since $1 + 2A > 0$, therefore

$$-\tfrac{1}{2} < A < 1 \tag{5.10}$$

In consequence, both the density and pressure are infinitely positive at $\tau = 0$, and they decrease monotonically as τ increases until the boundary $\tau = 2\bar{m}$ is reached. The positive boundary values ρ_b^*, p_b^* are given by:

$$8\pi\rho_b^* = \frac{1}{(2\bar{m})^2}, \qquad 8\pi p_b^* = \frac{1 + A}{1 - A} \cdot \frac{1}{(2\bar{m})^2} \tag{5.11}$$

Another general property of the density and pressure is obtained from the ratio ρ^*/p^* and (5.9). It follows that

$$\left.\begin{aligned} \rho^* &< p^*, & 0 &< A < 1, \\ \rho^* &= p^*, & A &= 0 \\ \rho^* &> p^*, & -\tfrac{1}{2} &< A < 0 \end{aligned}\right\} \tag{5.12}$$

The condition $\rho > p$ is usually accepted in a R-region. In contrast, T-regions may also have $\rho^* \leqslant p^*$.

The mass-function calculated from p^* in (5.9) and (4.12) is

$$2m^*(\tau) = C^* + \frac{\tau}{1 - A^2}\left\{\left(\frac{2\bar{m}}{\tau}\right)^{2(A+1)}\right\} - A^2 \tag{5.13}$$

and then (4.15) shows that $C^* = 0$. Hence the mass-function for the solution (5.9) is

$$2m^*(\tau) = \frac{1}{1 - A^2}\left\{\left(\frac{2\bar{m}}{\tau}\right)^{2(A+1)} - A^2\right\} \tag{5.14}$$

Case (b): $1 + 2A = 0$. In this case equations (5.2), (4.7), and (4.8) yield

$$\left.\begin{aligned} d\sigma^2 &= \frac{3}{4}\frac{d\tau^2}{2\bar{m}/\tau - 1} - \frac{dq^2}{\tau} - \tau^2\, d\Omega^2 \\ 8\pi\rho^* &= \frac{1}{\tau^2}, \qquad 8\pi p^* = \frac{1}{3\tau^2} \end{aligned}\right\} \tag{5.15}$$

The equation of state is therefore

$$p^* = \tfrac{1}{3}\rho^* \tag{5.16}$$

and thus the solution could be interpreted as representing a distribution of black-body radiation submerged within the horizon surface. Solutions of Einstein's equations satisfying the equation of state $p = \frac{1}{3}\rho$ also exist in an R-region and have been studied by Klein (1948) who showed that the distribution was of infinite spatial extent. It can be argued that this is not true for the T-solution (5.15). Let it be supposed that a photon can start from $q = 0$ at time $\tau = 0$ and travel unimpeded outwards along a radial null-geodesic. Its motion is therefore given by

$$\frac{dq}{d\tau} = \left(\frac{3}{4}\right)^{1/2} \frac{\tau}{(2\bar{m} - \tau)^{1/2}}$$

The appropriate integral of this equation is

$$q = \left(\frac{4}{3}\right)^{1/2} (2\bar{m})^{3/2} \left\{1 - \left(1 - \frac{\tau}{2\bar{m}}\right)^{1/2}\right\}^2 \left\{1 + \frac{1}{2}\left(1 - \frac{\tau}{2\bar{m}}\right)^{1/2}\right\}$$

Hence in $0 \leqslant \tau \leqslant 2\bar{m}$ there is no zero of q apart from $\tau = 0$; and, at $\tau = 2\bar{m}$, the value of q is finite. Thus the range of q over which the photon can travel possesses a finite upper limit, in contrast to the state of affairs in the vacuum T-region discussed in Section 2.

The mass-function calculated from p^* in (5.15) and (4.12) is

$$2m^*(\tau) = C^* - \tfrac{1}{3}\tau$$

and therefore (4.15) gives

$$2\bar{m} = C^* - \tfrac{1}{3}(2\bar{m})$$

and thus the mass-function for the solution (5.15) is

$$2m^*(\tau) = \tfrac{1}{3}(8\bar{m} - \tau)$$

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References

Cahill, M. E., and McVittie, G. C. (1970). *Journal of Mathematical Physics.*, **11**, 1382.
Carr, B. J., and Hawking, S. W. (1974). *Mon. Not. R. Astr. Soc.*, **168**, 399.
Klein, O. (1948). *Ark. F. Mat. Astron. o. Fysik*, **34A**, 1.
Korkina, M. P., and Gladush, V. D. (1972). *Ukr. Fiz. Zh.*, **17**, 1699.
Novikov, I. D. (1961). *Sov. Astron.*, **5**, 423.
Ruban, V. A. (1968). *Sov. Phys. JETP* (lett.), 8, 414.
Ruban, V. A. (1969). *Sov. Phys. JETP*, **29**, 1027.
Zeldovich, Y. D. and Novikov, I. D. (1971). *Relat. Astrophys. Vol. 1, Stars and Relativity*, p. 83, Chicago University Press, Chicago.